

Entanglement Creation and Storage in Two Qubits Coupling to an Anisotropic Heisenberg Spin Chain

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Abstract

The time evolution of the entanglement of a pair of two spin qubits is investigated when the two qubits simultaneously couple to an environment of an anisotropic Heisenberg XXZ spin chain. The entanglement of the two spin qubits can be created and is a periodic function of the time if the magnetic field is greater than a critical value. If the two spin qubits are in the Bell state, the entanglement can be stored with relatively large value even when the magnetic field is large.

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I. INTRODUCTION

Entangled quantum states are used mainly for quantum information processing, such as quantum teleportation, quantum secret-code and quantum computation [1–3]. Many investigations showed that entanglement exists naturally in the spin system when the temperature of the system is at zero [4–7]. In recent years, the study of the dynamics of entanglement [8–15] has attracted much attention in the manipulation of quantum systems. The dynamical properties and the time evolution of the entanglement in different quantum systems were investigated. These systems included mobile particle elastically-scattered by static spins [8], quantum mixed states [9, 10], two oscillators coupled to the same environment [11, 12], two d-level systems [13], decoherence of a spin-1/2 particle coupled to a spin bath in thermal equilibrium [14], a spin chain in driving the the decoherence of a coupled quantum system [15], etc. Meanwhile, the effects of the environment were taken into account. The excitation and quantum information transfer was investigated between two external spins when they coupled to a one-dimensional spin chain at different sites [16]. The entanglement induced by two external spins could be used to signal the critical points when they were simultaneously coupled to an environmental XY spin chain [17, 18]. The decay of the Loschmidt echo was enhanced by the quantum criticality of the surrounding Ising chain when an external spin was coupled to the environment [19]. When two external spins coupled to a transverse field Ising chain, the induced entanglement could be enhanced near quantum criticality and could be used to detect the quantum phase transition [20], which occurred in the many-body quantum systems [21]. The dynamical properties of the entanglement in a spin system need to be further investigated when it is coupled to an external environment.

In this paper, the dynamics and the time evolution of the entanglement of a pair of two qubits are investigated when the two qubits simultaneously couple to an environment of an anisotropic antiferromagnetic Heisenberg spin chain with magnetic field. In Section II, the Hamiltonian of the system and the effective Hamiltonian of the two qubits coupled to the environment are presented. In Section III, the time evolution of the system is analyzed for the simplest case of the environment. The entanglement creation in the coupled pair of two external qubits is discussed in Section IV. In Section V, the storage of the entanglement in the coupled pair of two external qubits is investigated. A discussion concludes the paper.

II. HAMILTONIAN OF THE SYSTEM

When two external spin qubits are coupled with the environment of a one-dimensional spin chain, the Hamiltonian of this system can be written as

$$H = H_0 + H_I. \quad (1)$$

where H_0 is the Hamiltonian of the environment. If the environment is an anisotropic Heisenberg XXZ spin chain, one has

$$H_0 = J \sum_{i=1}^N (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta_i \sigma_i^z \sigma_{i+1}^z) + B \sum_{i=1}^N \sigma_i^z, \quad (2)$$

where σ_i^α ($\alpha = x, y, z$) are Pauli operators, N is the number of the spin chain, J is the coupling coefficient between the spins, B is the magnetic field along the z -axis with the anisotropy $\Delta_i = \Delta \in (0,1)$. In Eq. (1), H_I is the interaction Hamiltonian between the two external spin qubits and the environment and can be written as,

$$H_I = J_p \sum_{i=1}^N (\sigma_a \sigma_i + \sigma_b \sigma_i). \quad (3)$$

where σ_a and σ_b are the Pauli operators of the qubits a and b , J_p is the coupling coefficient between the external spin qubits (a and b) and the Heisenberg spin chain. In order to facilitate the calculation, the coupling coefficients are chosen as $J = 1$ and $J_p = 0.2$ in this paper. That is, the environment is represented by the antiferromagnetic Heisenberg XXZ spin model. The schematic diagram of the system is shown in Fig. 1. The two qubits are symmetrically located at the two sides of the spin chain.

Fröhlich transformation [20, 22] can be used to solve the problem of induced effective interaction between two qubits through the medium of the Heisenberg spin chain. The environment of the antiferromagnetic Heisenberg spin chain has non-degenerate ground state $|\psi_0\rangle$ with ground state energy E_0 . According to the standard canonical transformation [20, 22, 23], the effective Hamiltonian of the external spin qubits can be written as

$$H_{eff}^{ab} = \sum_{j=1}^k \frac{\langle \psi_0 | H_i P_j H_i | \psi_0 \rangle}{E_j - E_0}, \quad (4)$$

where the projector is $P_j = |\psi_j\rangle\langle\psi_j|$ and $|\psi_j\rangle$ ($j = 1, 2, \dots, k$) is the time dependent excited state with energy E_j . After some straightforward calculations, the effective Hamiltonian can be reduced to

$$H_{eff}^{ab} = - \sum_j 2J_p J_p \sum_{\alpha, \beta} \Re(m_\alpha n_\beta^*) \sigma_a^\alpha \sigma_b^\beta + \sum_{\alpha, \beta} \frac{J_p^2}{4} (|m_\alpha|^2 + |n_\beta|^2), \quad (5)$$

where the parameters are $m_\alpha = \frac{\langle \psi_0 | s_m^\alpha | \psi_j \rangle}{\sqrt{E_k - E_0}}$, $n_\beta = \frac{\langle \psi_0 | s_n^\beta | \psi_j \rangle}{\sqrt{E_k - E_0}}$, $s^{\alpha, \beta} = \frac{1}{2} \sigma^{\alpha, \beta}$, and $\Re(m_\alpha n_\alpha^*)$ means the real part of the product $(m_\alpha n_\alpha^*)$ with $\alpha, \beta = x, y, z$. When the eigenstate $|\psi_j\rangle$ and the corresponding eigenvalue E_j of H_0 are obtained, the effective Hamiltonian H_{eff}^{ab} can be easily calculated.

III. ANALYSIS OF TIME EVOLUTION

In order to describe the time evolution of the entanglement of two-qubit system, the concurrence is used as a measure of the entanglement. The concurrence is defined as [24, 25]

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (6)$$

where the $\lambda_i (i = 1, 2, 3, 4)$ are the square roots of the eigenvalues of the density matrix ϱ_{ab} . The density matrix ϱ_{ab} is given by

$$\varrho_{ab} = \rho_{12}(\sigma_1^y \otimes \sigma_2^y) \rho_{12}^*(\sigma_1^y \otimes \sigma_2^y). \quad (7)$$

The ground state of the environment of the Heisenberg spin chain can be chosen as $|\phi_0\rangle$ while that of the two external spin qubits a and b can be chosen as $|01\rangle$. Under the influence of the environment, the two external spin qubits have an initial state as follows

$$|\psi_0\rangle = |\phi_0\rangle \otimes |01\rangle_{ab}. \quad (8)$$

The time evolution of the state is

$$|\psi(t)\rangle = \exp(-iH_{eff}^{ab}t)|\psi_0\rangle_{ab}, \quad (9)$$

with the density matrix $\varrho_{ab} = |\psi(t)\rangle\langle\psi(t)|$.

The reduced density matrix $\varrho_{ab}(t)$ can be written as

$$\varrho_{ab}(t) = \begin{pmatrix} u(t) & 0 & 0 & 0 \\ 0 & w_1(t) & y(t) & 0 \\ 0 & y^*(t) & w_2(t) & 0 \\ 0 & 0 & 0 & v(t) \end{pmatrix} \quad (10)$$

in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The corresponding concurrence $C(t)$ of the two external spin qubits can be calculated from the reduced density matrix $\varrho_{ab}(t)$ and given by

$$C(t) = 2 \max\{|y(t)| - \sqrt{u(t)v(t)}, 0\}. \quad (11)$$

IV. ENTANGLEMENT CREATION

For the simplest case of $N = 2$ in the anisotropic Heisenberg XXZ spin chain, the eigenenergies and eigenstates of the system are $E_1 = \Delta - 2B, E_2 = \Delta + 2B, E_3 = -\Delta + 2, E_4 = -\Delta - 2$ and $|\varphi_1\rangle = |11\rangle, |\varphi_2\rangle = |00\rangle, |\varphi_3\rangle = \frac{\sqrt{2}}{2}(|01\rangle + |10\rangle), |\varphi_4\rangle = \frac{\sqrt{2}}{2}(-|01\rangle + |10\rangle)$ respectively. When $B - \Delta > 1$, the ground state is $|\phi_0\rangle = |\varphi_1\rangle$. Then the effective Hamiltonian H_{eff}^{ab} can be written as

$$H_{eff}^{ab} = g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (12)$$

where the parameter g is given by $g = J_p^2 \frac{\Delta - B}{(B - \Delta + 1)(B - \Delta - 1)}$. The matrix elements $u(t), w_1(t), y(t), w_2(t)$ and $v(t)$ in Eq. (10) are given by $u(t) = 0, w_1(t) = \frac{1}{2} + \frac{1}{4}e^{-i4gt} + \frac{1}{4}e^{i4gt}, y(t) = -\frac{1}{4}e^{-i4gt} + \frac{1}{4}e^{i4gt}, w_2(t) = \frac{1}{2} - \frac{1}{4}e^{-i4gt} - \frac{1}{4}e^{i4gt}, v(t) = 0$. When $B - \Delta < 1$, the effective Hamiltonian $H_{eff}^{ab} = 0$. There is no entanglement between spin qubits a and b . When $B - \Delta = 1$, the ground state energy equals the excited state energy, i. e., $E_1 = E_4$. The energies of the two states are crossed at this point. Since the two states are degenerate, Eq. (4) is not valid to calculate the effective Hamiltonian H_{eff}^{ab} when $B - \Delta = 1$. That is, there is a critical value of the magnetic field B_C . The value of B_C is given by $B_C = 1 + \Delta$. If $B < B_C$, the concurrence $C(t)$ is zero. That is, there is no entanglement when $B < B_C$. If

$B > B_C$, the entanglement appears. That is, the entanglement can be created when $B > B_C$. Then the concurrence $C(t)$ can be given by

$$C(t) = \begin{cases} 0, & (B - \Delta < 1); \\ |\sin(4gt)|, & (B - \Delta > 1). \end{cases} \quad (13)$$

The concurrence $C(t)$ as a function of the time t is plotted in Fig. 2 when the magnetic field B and the anisotropy Δ are varied. The values of the anisotropy are $\Delta = 0.2, 0.4$ and 0.6 with $B > B_C$ in Figs. 2(a), 2(b) and 2(c) respectively. From Fig. 2, it is seen that the concurrence $C(t)$ is a periodic function of time t . It almost oscillates between the maximum value of one and the minimum value of zero. The period decreases as the magnetic field B increases.

The anisotropic antiferromagnetic Heisenberg XXZ model was used to investigate the order-to-disorder transition of the material Cs_2CoCl_4 [26]. For the material Cs_2CoCl_4 , the anisotropy is $\Delta = 0.25$. When the number of spins in the environment of the Heisenberg XXZ chain is greater than two, there is no approximate analytic solution of H_{eff}^{ab} and $C(t)$. To calculate $C(t)$, the numerical computation needs to be performed. In Fig. 3, the concurrence $C(t)$ is plotted as a function of time t when the spin numbers in the environment are $N = 4, 6$ and 8 . From Fig. 3, it is seen that the concurrence $C(t)$ is a periodic function of t with two different kinds of periods. Both periods decrease as the spin number N in the environment increase. There is a critical value B_C of the magnetic field. When $B < B_C$, the concurrence $C(t)$ oscillates following the large period. The period decreases slightly as B increases. While $B > B_C$, $C(t)$ oscillates following the small period. The period increases as B increases. The concurrence $C(t)$ can be approximately given by

$$C(t) \sim \begin{cases} |\sin[g(N)\sqrt{N/(N+1)}t]|, & (B < B_C); \\ |\sin[g(N)Nt]|, & (B > B_C). \end{cases} \quad (14)$$

Where $g(N)$ is a function of the spin number N in the environment. Though there is no analytic expression of the critical field B_C , it can be numerical calculated. The critical field B_C is plotted in Fig. 3(d) as a function of $1/N$. From Fig. 3(d), it is seen that B_C decreases linearly as $1/N$ decreases. In the thermodynamic limit of $N \rightarrow \infty$, B_C tends to zero. The regime for larger period of oscillation disappears.

V. ENTANGLEMENT STORAGE

The concurrence $C(t)$ is plotted as a function of magnetic field B and time t in Fig. 4 when the initial state of the two external spin qubits a and b is in the Bell state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The anisotropy is chosen as $\Delta = 0.25$ [26]. The spin numbers in the environment of the anisotropic antiferromagnetic Heisenberg chain are $N = 2, 4, 6$, and 8 . From Fig. 4, it is seen that the concurrence $C(t)$ is an oscillation function of time t . The oscillation period decreases as B increases. Obviously, the concurrence $C(t)$ is divided into several regions by different critical values of the magnetic field B_C . The red circles in Fig. 4 show the critical values of B_C . At the critical point B_C , the energies of the ground and excited states are crossing and the states are degenerate. For $N = 2$, there are two parts in $C(t)$ divided by one B_C . For $B < B_C$, $C(t)$ is almost a constant of $C(t) = 1.0$. For $B > B_C$, $C(t)$ oscillates with a small period [cf. Fig. 4(a)]. For $N = 4$, there are three parts divided by two different values of B_C [(marked by two red circles in Fig. 4(b))]. In the first part, $C(t)$ is very close to 1.0. It oscillates with quite small amplitude. In the second part, $C(t)$ oscillates with small period. In the third part, $C(t)$ oscillates with even smaller period [cf. Fig. 4(b)]. When $N = 6$ and 8 in Figs. 4(c) and 4(d), similar phenomena occurs. Obviously, the concurrence $C(t)$ is divided into $(N/2 + 1)$ parts by $N/2$ critical values of B_C . The energy is crossing at the critical values of B_C in the ground state as well as in excited states. The concurrence $C(t)$ is jumping as the state is switched from an entangled state to another. In the thermodynamic limit, the continuous energy level crossings occur [27]. The first part of concurrence $C(t)$ disappears. Other parts of $C(t)$ tends to smooth and continuous. In Fig. 3(d), only the first critical value of B_C as a function of spin number $1/N$ is plotted. From Fig. 4, it is also clear that the entanglement $C(t)$ can keep large value even for relatively large magnetic field B . If the initial state of the two external spin qubits is the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, similar results as that shown in Fig. 4 are obtained.

VI. DISCUSSION

The time evolution of the entanglement of two external spin qubits is investigated when they are coupled to the environment of an anisotropic antiferromagnetic Heisenberg XXZ spin chain with magnetic field. The approximate form of the effective Hamiltonian is derived.

The concurrence is used as a measure of the entanglement. When there are two spins in the environment, there is no entanglement between two external spin qubits when the magnetic field is smaller than a critical value. When the magnetic field is greater than the critical value, the entanglement can be created and is a periodic function of the time. The entanglement almost oscillates between one and zero. The oscillation period decrease as the anisotropy and the magnetic field increase. There are $N/2 + 1$ parts in the entanglement divided by $N/2$ values of critical magnetic fields. The first critical magnetic field tends to zero when the spin number in the environment tends to infinity. When the initial state of the two external spin qubits is in one of the Bell state, the entanglement can be stored. Though there are different regimes in the entanglement, the entanglement always keeps quite large value when it oscillates with increasing number of spins in the environment.

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Figure Captions

Fig. 1

The schematic diagram of two external spin qubits symmetrically coupled to the environment of an anisotropic Heisenberg spin chain.

Fig. 2

The concurrence $C(t)$ is plotted as a function of the time t for $N = 2$ when the magnetic field B and the anisotropy Δ are varied with $B > B_C$. (a). $\Delta = 0.2$. (b). $\Delta = 0.4$. (c). $\Delta = 0.8$.

Fig. 3

The concurrence $C(t)$ is plotted as a function of the magnetic field B and the time t for different spin numbers N of the environment in (a), (b), and (c). The anisotropy is $\Delta = 0.25$. (a). $N = 4$. (b). $N = 6$. (c). $N = 8$. (d). The critical field B_C is plotted as a function of $1/N$.

Fig. 4

The concurrence C is plotted as a function of the magnetic field B and the time t for different spin numbers in the environment. The anisotropy is $\Delta = 0.25$ and the Bell state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is chosen. (a). $N = 2$. (b). $N = 4$. (c). $N = 6$. (d). $N = 8$.

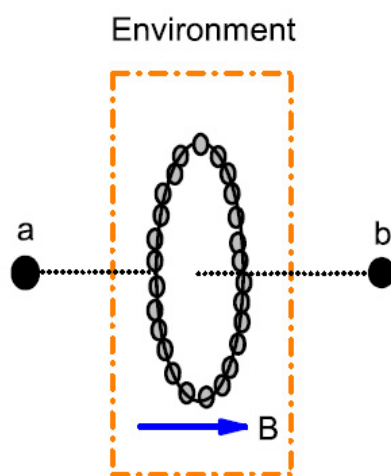


Fig. 1

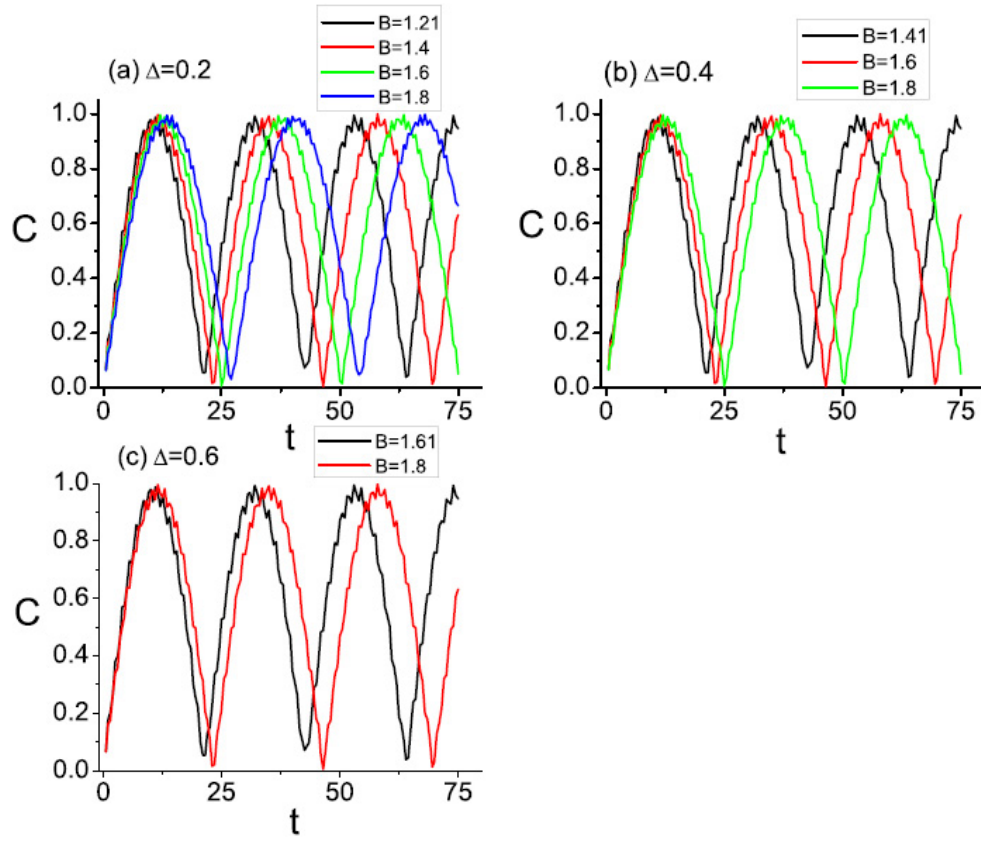


Fig. 2

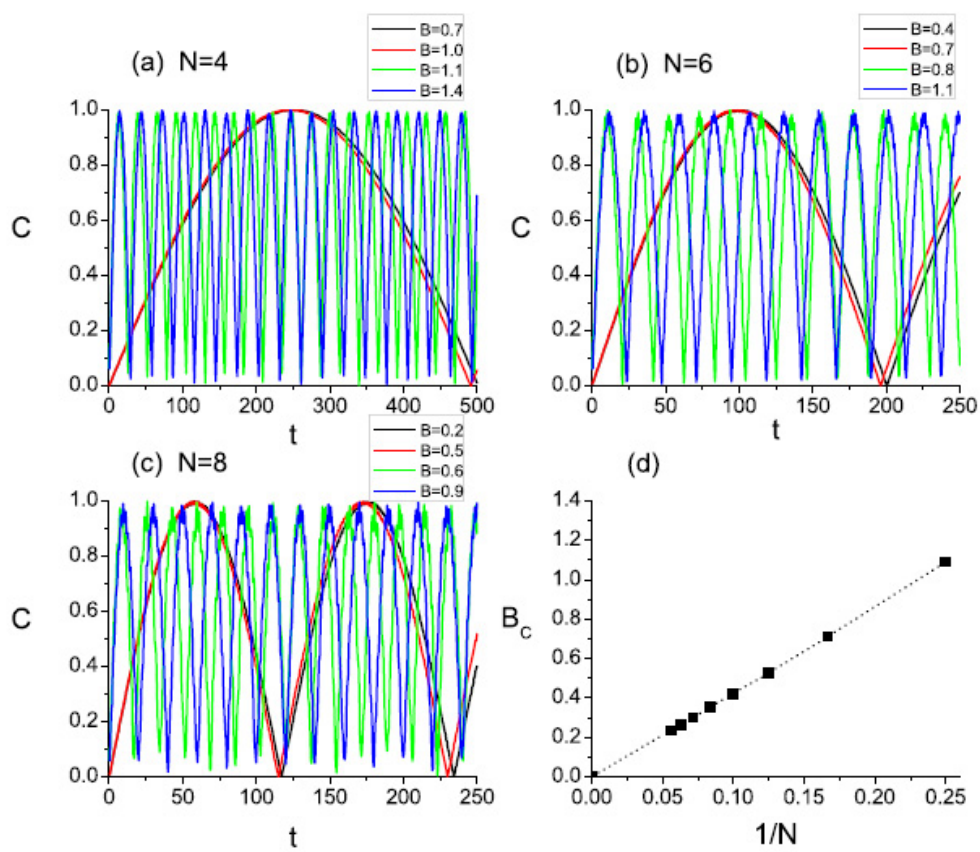


Fig. 3

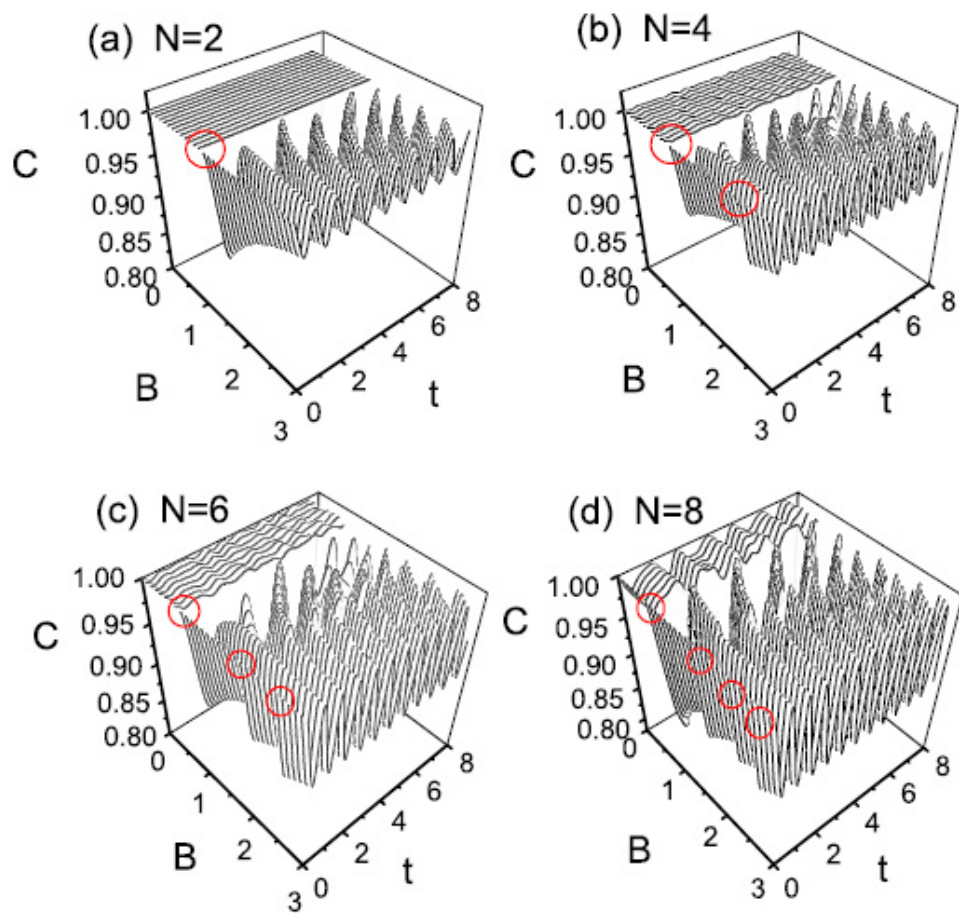


Fig. 4